

# Born-Infeld Black Holes in (A)dS Spaces

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We study some exact solutions in a  $D(\geq 4)$ -dimensional Einstein-Born-Infeld theory with a cosmological constant. These solutions are asymptotically de Sitter or anti-de Sitter, depending on the sign of the cosmological constant. Black hole horizon and cosmological horizon in these spacetimes can be a positive, zero or negative constant curvature hypersurface. We discuss the thermodynamics associated with black hole horizon and cosmological horizon. In particular we find that for the Born-Infeld black holes with Ricci flat or hyperbolic horizon in AdS space, they are always thermodynamically stable, and that for the case with a positive constant curvature, there is a critical value for the Born-Infeld parameter, above which the black hole is also always thermodynamically stable, and below which a unstable black hole phase appears. In addition, we show that although the Born-Infeld electrodynamics is non-linear, both black hole horizon entropy and cosmological horizon entropy can be expressed in terms of the Cardy-Verlinde formula. We also find a factorized solution in the Einstein-Born-Infeld theory, which is a direct product of two constant curvature spaces: one is a two-dimensional de Sitter or anti-de Sitter space, the other is a  $(D-2)$ -dimensional positive, zero or negative constant curvature space.

## I. INTRODUCTION

Black holes in anti-de Sitter (AdS) and de Sitter (dS) spaces are quite different from their counterparts in asymptotically flat spaces. In AdS spaces, there are so-called topological black holes whose event horizon could be a positive, zero or negative constant curvature surface [1]. In asymptotically dS spaces, there are no globally time-like Killing vector and spatial infinity, there is still not a well defined approach to define conserved charges like mass and angular momentum of asymptotically dS gravitational configurations [2, 3, 4]. Due to the AdS/CFT correspondence [5], it was argued by Witten [6] that the thermodynamics of black holes in AdS spaces can be identified with that of dual conformal field theory (CFT) residing on the boundary of the AdS space. In the sense of the dS/CFT correspondence [7], thermodynamics of black hole horizon and of cosmological horizon in the asymptotically de Sitter spaces might be related to that of dual CFTs. These motivate a surge of study of black holes in AdS and dS spaces.

For a  $(1+1)$ -dimensional conformal field theory, there is a well-known entropy formula, namely, the Cardy formula [8]. In an elegant paper [9], in the spirit of AdS/CFT correspondence, Verlinde argued that there is a similar entropy formula for CFTs in higher dimensions. The formula “derived” by Verlinde is called Cardy-Verlinde formula in the literature. Indeed this formula has been checked to hold for CFTs with AdS gravity duals, such as Schwarzschild-AdS black holes [9], Kerr-AdS black holes [10], Hyperbolic and charged black holes [11], Taub-Bolt-AdS instanton [12], Kerr-Newmann-AdS black holes [13] and so on (see also [14]). In addition, it has been found that entropies of black hole horizons [15] and cosmological horizons [16] in asymptotically dS spaces can also be expressed in terms of the Cardy-Verlinde formula.

In 1934 Born and Infeld proposed a non-linear electrodynamics with the aim of obtaining a finite value for the self-energy of a point-like charge [17]. Although it became less popular with the introduction of QED, in recent years the Born-Infeld action has been occurring repeatedly with the development of superstring theory, where the dynamics of D-branes, some soliton solutions of supergravity, is governed by the Born-Infeld action. For various motivations, extending the Reissner-Nordström black hole solutions in Einstein-Maxwell theory to the charged black hole solutions

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in Einstein-Born-Infeld theory with/without a cosmological constant has attracted some attention in recent years, for example, see [18, 19].

In this paper, we first generalize the exact solutions of spherically symmetric Born-Infeld black holes with a cosmological horizon in arbitrary dimensions recently given in [19] to the case where black hole horizon and/or cosmological horizon is a positive, zero or negative constant curvature surface. We then in Secs. III and IV study thermodynamics associated with black hole horizon and cosmological horizon and show that although the electrodynamics is non-linear, entropies of black holes and cosmological horizon in the Einstein-Born-Infeld theory can still be expressed in terms of the Cardy-Verlinde formula, which shows the latter is of some universality. In Sec. V we present a factorized solution in the Einstein-Born-Infeld theory. The solution is a direct product of two constant curvature spaces: one is a two-dimensional dS or AdS space and the other is a positive, zero or negative constant curvature space. The result is summarized in Sec. VI.

## II. BORN-INFELD BLACK HOLES IN (A)DS SPACES

Consider an  $(n+1)$ -dimensional ( $n \geq 3$ ) Einstein-Born-Infeld action with a cosmological constant  $\Lambda$

$$S = \int d^{n+1}x \sqrt{-g} \left( \frac{\mathcal{R} - 2\Lambda}{16\pi G} + L(F) \right), \quad (2.1)$$

where  $\mathcal{R}$  is scalar curvature and  $L(F)$  is given by

$$L(F) = 4\beta^2 \left( 1 - \sqrt{1 + \frac{F^{\mu\nu}F_{\mu\nu}}{2\beta^2}} \right). \quad (2.2)$$

Here the constant  $\beta$  is called Born-Infeld parameter with dimension of mass. In the limit  $\beta \rightarrow \infty$ ,  $L(F)$  reduces to the standard Maxwell form

$$L(F) = -F^{\mu\nu}F_{\mu\nu} + \mathcal{O}(F^4), \quad (2.3)$$

while one has  $L(F) \rightarrow 0$  as  $\beta \rightarrow 0$ . In what follows we set  $16\pi G = 1$  for simplicity, where  $G$  is the Newton constant in  $(n+1)$ -dimensions.

The equations of motion of the electromagnetic field and the Einstein equations can be obtained by varying the action with respect to the gauge field  $A_\mu$  and the metric  $g_{\mu\nu}$ , which yields

$$\partial_\mu \left( \frac{\sqrt{-g}F^{\mu\nu}}{\sqrt{1 + \frac{F^{\mu\nu}F_{\mu\nu}}{2\beta^2}}} \right) = 0, \quad (2.4)$$

$$\mathcal{R}_{\mu\nu} - \frac{1}{2}\mathcal{R}g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{1}{2}g_{\mu\nu}L(F) + \frac{2F_{\mu\alpha}F_\nu{}^\alpha}{\sqrt{1 + \frac{F^{\mu\nu}F_{\mu\nu}}{2\beta^2}}}, \quad (2.5)$$

respectively. Here  $\mathcal{R}_{\mu\nu}$  stands for Ricci tensor.

Suppose the spacetime metric is of the form

$$ds^2 = -V(r)dt^2 + \frac{1}{V(r)}dr^2 + R^2(r)h_{ij}dx^i dx^j, \quad (2.6)$$

where  $V(r)$  and  $R(r)$  are two functions of the coordinate  $r$  only, and  $h_{ij}$  is a function of coordinates  $x^i$  which span an  $(n-1)$ -dimensional hypersurface with constant scalar curvature  $(n-1)(n-2)k$ . Here  $k$  is a constant and characterizes the hypersurface. Without loss of generality, one can take  $k$  to be  $\pm 1$  and  $0$  such that the black hole horizon or cosmological horizon in (2.6) can be a positive (elliptic), zero (flat) or negative (hyperbolic) constant curvature hypersurface. For the metric (2.6), we have non-vanishing components of Ricci tensor

$$\mathcal{R}_t^t = -\frac{V''}{2} - (n-1)\frac{V'R'}{2R}, \quad (2.7)$$

$$\mathcal{R}_r^r = -\frac{V''}{2} - (n-1)\frac{V'R'}{2R} - (n-1)\frac{VR''}{R}, \quad (2.8)$$

$$\mathcal{R}_j^i = \left( \frac{n-2}{R^2}k - \frac{1}{(n-1)R^{n-1}}[V(R^{n-1})']' \right) \delta_j^i, \quad (2.9)$$

where a prime stands for the derivative with respect to the coordinate  $r$ .

In the static and symmetric background (2.6), the equation (2.4) can be satisfied by setting  $F^{\mu\nu}$  to zero, except for  $F^{rt}$ , which gives

$$F^{rt} = \frac{\sqrt{(n-1)(n-2)}\beta q}{\sqrt{2\beta^2 R^{2n-2} + (n-1)(n-2)q^2}}, \quad (2.10)$$

where  $q$  is an integration constant relating to the electric charge of the solution. Defining the electric charge via  $Q = \frac{1}{4\pi} \int {}^*F d\Omega$ , we have

$$Q = \frac{\sqrt{(n-1)(n-2)}\omega_{n-1}}{4\pi\sqrt{2}}q, \quad (2.11)$$

where  $\omega_{n-1}$  represents the volume of constant curvature hypersurface described by  $h_{ij}dx^i dx^j$ .

From Eqs. (2.7) and (2.8), we have  $R''(r) = 0$ , which has two solutions. One is  $R = r$ ; the other is  $R = a$ , here  $a$  is a constant. We first consider the case of  $R = r$ . The case of  $R = a$  will be discussed later. With  $R = r$ , solving the equation (2.9) yields

$$\begin{aligned} V(r) = & k - \frac{m}{r^{n-2}} + \left( \frac{4\beta^2}{n(n-1)} + \frac{1}{l^2} \right) r^2 \\ & - \frac{2\sqrt{2}\beta}{(n-1)r^{n-2}} \int \sqrt{2\beta^2 r^{2n-2} + (n-1)(n-2)q^2} dr. \end{aligned} \quad (2.12)$$

Here we have redefined the cosmological constant as  $\Lambda = -n(n-1)/2l^2$ . It turns out that the integration in (2.12) can be worked out and can be expressed by using a hypergeometric function,

$$\begin{aligned} V(r) = & k - \frac{m}{r^{n-2}} + \left( \frac{4\beta^2}{n(n-1)} + \frac{1}{l^2} \right) r^2 \\ & - \frac{2\sqrt{2}\beta}{n(n-1)r^{n-3}} \sqrt{2\beta^2 r^{2n-2} + (n-1)(n-2)q^2} \\ & + \frac{2(n-1)q^2}{nr^{2n-4}} {}_2F_1\left[\frac{n-2}{2(n-1)}, \frac{1}{2}, \frac{3n-4}{2(n-1)}, -\frac{(n-1)(n-2)q^2}{2\beta^2 r^{2n-2}}\right]. \end{aligned} \quad (2.13)$$

Here  $m$  is an integration constant, relating to the mass of the gravitational configuration. Since the metric is asymptotically de Sitter ( $l^2 < 0$ ) or anti-de Sitter ( $l^2 > 0$ ), according to the definition of mass in asymptotically dS and AdS spaces due to Abbott and Deser [2], we have the mass of the solution

$$M = (n-1)\omega_{n-1}m. \quad (2.14)$$

Note that throughout this paper, the convention  $16\pi G = 1$  has been used. When  $k = 1$  and  $h_{ij}dx^i dx^j$  denotes the line element of an  $(n-1)$ -dimensional unit round sphere, the solution (2.13) reduces to the one found in [19]. Note that for a positive constant curvature space, the metric is not necessary to be a round sphere. Therefore the metric (2.6) is more general than a spherically symmetric metric.

### III. CARDY-VERLINDE FORMULA FOR THE BORN-INFELD BLACK HOLES IN ADS SPACES

In this section we first consider the case of  $l^2 > 0$ , namely for a negative cosmological constant. In this case, the spacetime asymptotically approaches to an AdS space. The solution (2.13) describes a Born-Infeld black hole in AdS space. The black hole horizon is determined by  $V(r)|_{r=r_+} = 0$ . The behavior of metric function  $V$  in the small  $r$  region and large  $r$  region and thermodynamics of the black hole for the case of  $k = 1$  have been analyzed and discussed in [19]. We will not therefore repeat them here, instead we will mainly focus on the cases of  $k = 0$  and  $k = -1$  and will show that the black hole entropy can be expressed in terms of the Cardy-Verlinde formula.

The temperature of the black hole can be obtained by continuing the metric (2.6) to its Euclidean sector via  $t = -i\tau$  and requiring the absence of conical singularity at the horizon. This results in a periodic Euclidean time  $\tau$  with period  $1/T$ , which is just the inverse Hawking temperature of the black hole. Calculation gives

$$T = \frac{1}{4\pi r_+} \left( (n-2)k + \left( \frac{4\beta^2}{n-1} + \frac{n}{l^2} \right) r_+^2 - \frac{2\sqrt{2}\beta}{(n-1)r_+^{n-3}} \sqrt{2\beta^2 r_+^{2n-2} + (n-1)(n-2)q^2} \right). \quad (3.1)$$

When  $T = 0$ , the black hole is an extremal black hole, there the black hole horizon and the internal horizon (Cauchy horizon) coincide with each other. For the extremal black hole, the charge has a relation to the horizon radius

$$(n-1)(n-2)q^2 = -2\beta^2 r_+^{2n-2} + \frac{(n-1)^2 r_+^{2n-6}}{8\beta^2} \left( (n-2)k + \left( \frac{4\beta^2}{n-1} + \frac{n}{l^2} \right) r_+^2 \right)^2. \quad (3.2)$$

The entropy of the black hole still obeys the so-called horizon area formula

$$S = 4\pi\omega_{n-1}r_+^{n-1}. \quad (3.3)$$

It is easy to show that these thermodynamic quantities such as charge (2.11), mass (2.14), temperature (3.1) and entropy (3.3) satisfy the first law of black hole thermodynamics

$$dM = TdS + \Phi dQ, \quad (3.4)$$

where  $\Phi$  is the electrostatic potential at the black hole horizon, which is conjugate to the electric charge  $Q$ ,

$$\Phi = \sqrt{\frac{n-1}{2(n-2)}} \frac{16\pi q}{r_+^{n-2}} {}_2F_1\left[\frac{n-2}{2(n-1)}, \frac{1}{2}, \frac{3n-4}{2(n-1)}, -\frac{(n-1)(n-2)q^2}{2\beta^2 r_+^{2n-2}}\right]. \quad (3.5)$$

Note that the mass of black hole can be expressed in terms of the horizon radius  $r_+$

$$\begin{aligned} M = & (n-1)\omega_{n-1}r_+^{n-2} \left( k + \left( \frac{4\beta^2}{n(n-1)} + \frac{1}{l^2} \right) r_+^2 \right. \\ & - \frac{2\sqrt{2}\beta}{n(n-1)r_+^{n-3}} \sqrt{2\beta^2 r_+^{2n-2} + (n-1)(n-2)q^2} \\ & \left. + \frac{2(n-1)q^2}{nr_+^{2n-4}} {}_2F_1\left[\frac{n-2}{2(n-1)}, \frac{1}{2}, \frac{3n-4}{2(n-1)}, -\frac{(n-1)(n-2)q^2}{2\beta^2 r_+^{2n-2}}\right] \right). \end{aligned} \quad (3.6)$$

To see the thermodynamical stability of the black hole, let us calculate the heat capacity with a fixed charge, which implies that we are discussing the stability in a canonical ensemble.

$$\begin{aligned} C_q &= \left( \frac{\partial M}{\partial T} \right)_q \\ &= \frac{16\pi^2 r_+^n T}{-(n-2)k + \left( \frac{4\beta^2}{n-1} + \frac{n}{l^2} \right) r_+^2 + \frac{2\sqrt{2}\beta}{(n-1)r_+^{n-3}} \frac{-2\beta^2 r_+^{2n-2} + (n-1)(n-2)q^2}{\sqrt{2\beta^2 r_+^{2n-2} + (n-1)(n-2)q^2}}}. \end{aligned} \quad (3.7)$$

Note that from the metric (2.13), Hawking temperature (3.1) or the heat capacity (3.7), one can see that the case of  $\beta = 0$  or  $q = 0$  reduces to the case of Schwarzschild black hole in AdS space. In other words, when  $\beta = 0$ , the charge parameter  $q$  will disappear automatically, and vice versa. When  $q = 0$ , we can see that the heat capacity is always positive if  $k = 0$  or  $-1$ , while it is negative for  $r_+ < l\sqrt{(n-2)/n}$ , positive for  $r_+ > l\sqrt{(n-2)/n}$ , and diverges at  $r_+ = l\sqrt{(n-2)/n}$ , when  $k = 1$ . When  $q \neq 0$ , since the numerator in (3.7) is always positive, the sign of  $C_q$  is therefore completely determined by that of the denominator, from which we are certainly able to get a relation of the horizon radius to other parameters  $k$ ,  $\beta$ , spacetime dimension  $n$  and charge  $q$ . Numerical check indicates that when  $k = 0$  or  $-1$ , the heat capacity is always positive. This can be seen from the behavior of Hawking temperature (3.1). In Fig. 1 and Fig. 2, we respectively plot the inverse Hawking temperature for a given charge when  $k = 0$  and  $k = -1$ . We see that the inverse Hawking temperature always monotonically decreases from infinity (corresponding to extremal black holes) to zero as the black hole horizon radius increases. Note that the region with a negative Hawking temperature should be excluded from the physical phase space.

When  $k = 1$ , the heat capacity can be positive or negative. We plot the inverse Hawking temperature of the Born-Infeld black hole when  $k = 1$  in Fig. 3. In the case of  $k = 1$ , we notice that between two stable black hole phases (small black holes near the extremal limit and large black holes where the effect of charge is negligible) there is an unstable black hole phase. However, this unstable black hole phase will disappear when the Born-Infeld parameter  $\beta$  increases. To clearly see this point, in Fig. 4 we plot the inverse temperatures of black holes with a fixed charge, but with different  $\beta$ , from which we can see clearly how the unstable phase disappears. This implies that for the Born-Infeld black holes with a given charge, there is a critical value of the Born-Infeld parameter  $\beta$ , above which the

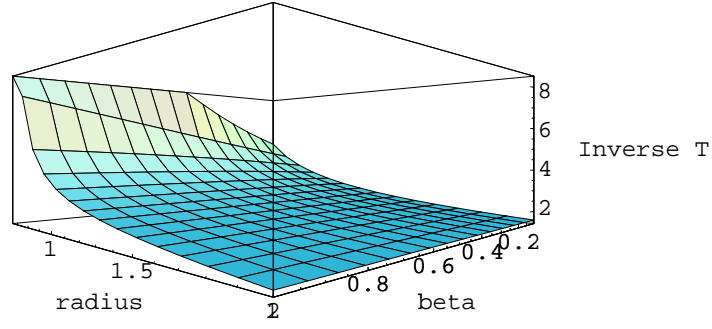


FIG. 1: The inverse Hawking temperature of a five-dimensional Born-Infeld black hole in AdS space for a fixed charge  $q/l^2 = 0.5$  for the case of  $k = 0$ .

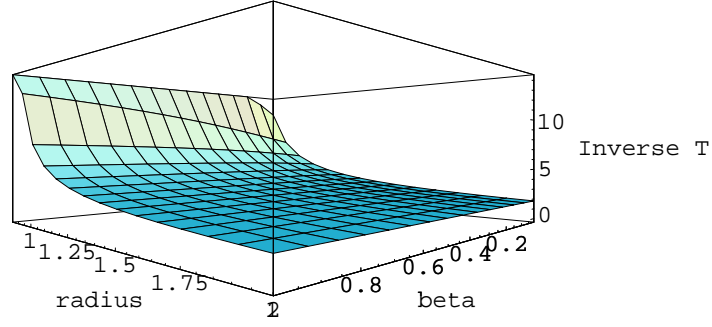


FIG. 2: The inverse Hawking temperature of a five-dimensional Born-Infeld black hole in AdS space for a fixed charge  $q/l^2 = 0.5$  for the case of  $k = -1$ .

black holes are always stable thermodynamically. This critical point can be calculated from the Hawking temperature (3.1). The expression, however, is a little bit complicated, we do not therefore present it here.

According to the AdS/CFT correspondence, the thermodynamics of Born-Infeld black holes in AdS spaces should also be identified with that of a some dual CFT. Since the entropy of CFTs with AdS duals can be described by the Cardy-Verlinde formula [9], so it is of interest to see whether or not the black hole entropy (3.3) can be recast to a form of the Cardy-Verlinde formula.

Suppose there is a CFT residing on an  $(n - 1)$ -dimensional sphere with radius  $R$ ,

$$ds^2 = -dt^2 + R^2 d\Omega_{n-1}^2. \quad (3.8)$$

The Cardy-Verlinde formula can be written down as [9]

$$S = \frac{2\pi R}{n-1} \sqrt{E_c(2E - E_c)}, \quad (3.9)$$

where  $E$  and  $E_c$  respectively represent the total energy and the Casimir energy, non-extensive part of energy, of the CFT.

For the Born-Infeld black holes, the total energy  $E$  is the AD mass  $M$  given in (2.14). The corresponding Casimir energy  $E_c$  is [9, 11]

$$E_c = nE - (n-1)TS - (n-1)\Phi Q. \quad (3.10)$$

Substituting those thermodynamic quantities into (3.10), we get

$$E_c = 2k(n-1)r_+^{n-2}\omega_{n-1}. \quad (3.11)$$

We see that for Born-Infeld black holes with Ricci flat horizon ( $k = 0$ ), the Casimir energy vanishes, while it is negative for the hyperbolic horizon.

With the energy  $E$  and the Casimir energy  $E_c$ , we find that the black hole entropy (3.3) can indeed be cast to a form of the Cardy-Verlinde formula as follows,

$$S = \frac{2\pi l}{n-1} \sqrt{(2(E - E_q) - E_c)E_c/k}. \quad (3.12)$$

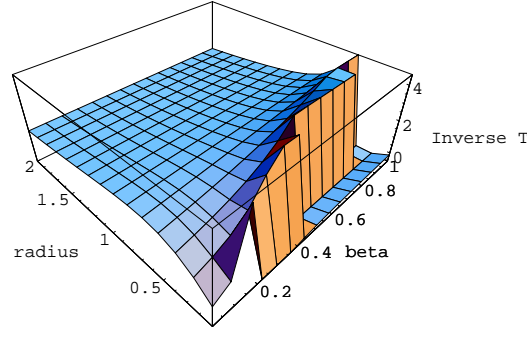


FIG. 3: The inverse Hawking temperature of a five-dimensional Born-Infeld black hole in AdS space for a fixed charge  $q/l^2 = 0.5$  for the case of  $k = 1$ .

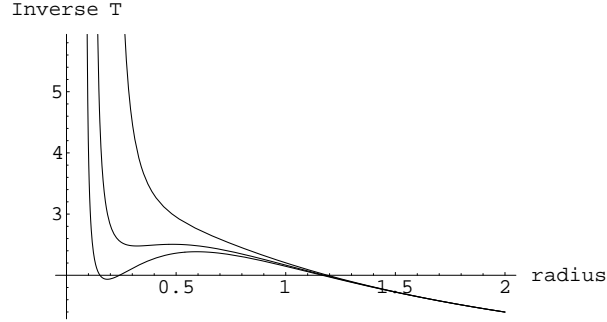


FIG. 4: The inverse Hawking temperature of a five-dimensional Born-Infeld black hole in AdS space for a fixed charge  $q/l^2 = 0.5$  for the case of  $k = 1$ . Three curves from right to left correspond to the case  $\beta l = 0.40, 0.22$  and  $0.15$ .

The dual CFT to the Born-Infeld black holes resides on the boundary of the bulk metric (2.6). Up to a conformal factor, the boundary metric is

$$ds^2 = -dt^2 + l^2 h_{ij} dx^i dx^j, \quad (3.13)$$

it is just the boundary spacetime, in which the dual CFT resides and its entropy is described by the formula (3.12).  $E_q$  in (3.12) is nothing but the energy of electromagnetic field outside the black hole horizon, which can be calculated as

$$\begin{aligned} E_q &\equiv \omega_{n-1} \int_{r_+}^{\infty} \bar{T}_0^0 r^{n-1} dr \\ &= \omega_{n-1} \left( \frac{4\beta^2}{n} r_+^n - \frac{2\sqrt{2}\beta r_+}{n} \sqrt{2\beta^2 r_+^{2n-2} + (n-1)(n-2)q^2} \right. \\ &\quad \left. + \frac{2(n-1)^2 q^2}{nr_+^{n-2}} {}_2F_1\left[\frac{n-2}{2(n-1)}, \frac{1}{2}, \frac{3n-4}{2(n-1)}, -\frac{(n-1)(n-2)q^2}{2\beta^2 r_+^{2n-2}}\right] \right), \end{aligned} \quad (3.14)$$

where  $\bar{T}_0^0$  is the  $0-0$  component of the energy-momentum tensor of non-linear electromagnetic field,

$$\bar{T}_{\mu\nu} = \frac{1}{2} g_{\mu\nu} L(F) + \frac{2F_{\mu\alpha} F_{\nu}^{\alpha}}{\sqrt{1 + \frac{F^{\mu\nu} F_{\mu\nu}}{2\beta^2}}}. \quad (3.15)$$

Thus we show that although the electrodynamics in the Einstein-Born-Infeld theory is non-linear, the entropy of the Born-Infeld black holes in AdS space can still be expressed in terms of the Cardy-Verlinde formula. Of course, the energy of electromagnetic field outside the black hole horizon should be subtracted from the total energy.

#### IV. CARDY-VERLINDE FORMULA FOR THE BORN-INFELD BLACK HOLES IN DS SPACES

In this section we consider the Born-Infeld black holes in de Sitter space ( $l^2 < 0$  in (2.12)). In this case, a cosmological horizon appears, in addition to black hole horizons. Both cosmological horizon and black hole horizon have thermodynamic properties such as Hawking temperature, entropy and etc. As we mentioned in Sec. I, it is not an easy matter to compute conserved charges associated with an asymptotically dS space because of the absence of spatial infinity and a globally timelike Killing vector in such a spacetime. It is found that if one uses the definition of mass due to Abbott and Deser in dS space [2], the entropy of Schwarzschild black holes in dS space can be expressed by the Cardy-Verlinde formula, but not for the entropy of cosmological horizon [15]. On the other hand, if one uses the definition of mass coming from the surface counterterm prescription in asymptotically dS spaces [3, 4], the entropy of cosmological horizon can be described by the Cardy-Verlinde formula, but not for the entropy of black hole horizon [16].

For convenience, we change  $l^2$  in (2.13) to  $-l^2$  so that in what follows one has  $l^2 > 0$ . In this case, the metric function  $V$  becomes

$$\begin{aligned} V(r) = & k - \frac{m}{r^{n-2}} + \left( \frac{4\beta^2}{n(n-1)} - \frac{1}{l^2} \right) r^2 \\ & - \frac{2\sqrt{2}\beta}{n(n-1)r^{n-3}} \sqrt{2\beta^2 r^{2n-2} + (n-1)(n-2)q^2} \\ & + \frac{2(n-1)q^2}{nr^{2n-4}} {}_2F_1\left[\frac{n-2}{2(n-1)}, \frac{1}{2}, \frac{3n-4}{2(n-1)}, -\frac{(n-1)(n-2)q^2}{2\beta^2 r^{2n-2}}\right]. \end{aligned} \quad (4.1)$$

The solution (4.1) is asymptotically de Sitter. When  $k \neq 1$ , the solution (4.1) generalizes the so-called topological de Sitter solutions proposed in [4] to the Einstein-Born-Infeld theory with a positive cosmological constant. When  $k = 1$  and  $m > 0$ , in an appropriate parameter space, both cosmological horizon  $r_c$  and black hole horizon  $r_+$  occur; they satisfy  $V(r)|_{r=r_+, r_c} = 0$ . When  $k = 0$  or  $k = -1$ , black hole horizon disappears, but the cosmological horizon is still there if  $m < 0$ . As the case of asymptotic AdS, the black hole horizon or cosmological horizon can be a positive, zero or negative constant curvature hypersurface. Therefore these solutions can be dubbed topologically asymptotic de Sitter solutions.

According to the definition of mass due to Abbott and Deser [2], the mass of the solution (4.1) is

$$M = (n-1)\omega_{n-1}m, \quad (4.2)$$

where  $m$  can be expressed in terms of black hole horizon radius  $r_+$  via  $V(r_+) = 0$ ,

$$\begin{aligned} m = & r_+^{n-2} \left( k + \left( \frac{4\beta^2}{n(n-1)} - \frac{1}{l^2} \right) r_+^2 \right. \\ & - \frac{2\sqrt{2}\beta}{n(n-1)r_+^{n-3}} \sqrt{2\beta^2 r_+^{2n-2} + (n-1)(n-2)q^2} \\ & \left. + \frac{2(n-1)q^2}{nr_+^{2(n-2)}} {}_2F_1\left[\frac{n-2}{2(n-1)}, \frac{1}{2}, \frac{3n-4}{2(n-1)}, -\frac{(n-1)(n-2)q^2}{2\beta^2 r_+^{2n-2}}\right] \right). \end{aligned} \quad (4.3)$$

The Hawking temperature of black hole horizon is found to be

$$T = \frac{1}{4\pi r_+} \left( (n-2)k + \left( \frac{4\beta^2}{n-1} - \frac{n}{l^2} \right) r_+^2 - \frac{2\sqrt{2}\beta}{(n-1)r_+^{n-3}} \sqrt{2\beta^2 r_+^{2n-2} + (n-1)(n-2)q^2} \right). \quad (4.4)$$

The entropy of black hole horizon and the chemical potential conjugate to the charge  $Q$  are still given by (3.3) and (3.5), respectively. Namely they keep the same forms as in the case of asymptotically AdS space. In this way it is easy to check that the first law of thermodynamics holds for the black hole horizon,  $dM = TdS + \Phi dQ$ .

For the black hole horizon, we find that the Casimir energy  $E_c$  is given by [15]

$$E_c = nE - (n-1)TS - (n-1)\Phi Q = 2k(n-1)r_+^{n-2}\omega_{n-1}, \quad (4.5)$$

and that the energy  $E_q$  of the electromagnetic field outside the black hole horizon remains the same form as the one given in (3.14). Here the total energy  $E = M$ .

Then the entropy of black hole horizon given by (3.3) can be recast to

$$S = \frac{2\pi l}{n-1} \sqrt{E_c/k|(2(E-E_q)-E_c)|}. \quad (4.6)$$

It is interesting to note that for the black hole in dS space, the extensive part of total energy

$$2(E-E_q)-E_c = -2(n-1)\omega_{n-1}r_+^n/l^2, \quad (4.7)$$

is negative. Its implication is not clear to us at the moment.

Next we discuss the thermodynamics of cosmological horizon  $r_c$ . We relate the gravitational mass  $\tilde{M}$  to the cosmological horizon  $r_c$  by using the surface counterterm method [3, 4]. In this method, the gravitational mass is just the AD mass, but with an opposite sign,

$$\tilde{M} = -(n-1)\omega_{n-1}m, \quad (4.8)$$

where  $m$  can be expressed in terms of the cosmological horizon radius  $r_c$

$$\begin{aligned} m = & r_c^{n-2} \left( k + \left( \frac{4\beta^2}{n(n-1)} - \frac{1}{l^2} \right) r_c^2 \right. \\ & - \frac{2\sqrt{2}\beta}{n(n-1)r_c^{n-3}} \sqrt{2\beta^2 r_c^{2n-2} + (n-1)(n-2)q^2} \\ & \left. + \frac{2(n-1)q^2}{nr_c^{2(n-2)}} {}_2F_1\left[\frac{n-2}{2(n-1)}, \frac{1}{2}, \frac{3n-4}{2(n-1)}, -\frac{(n-1)(n-2)q^2}{2\beta^2 r_c^{2n-2}}\right] \right). \end{aligned} \quad (4.9)$$

Note that the gravitational mass  $\tilde{M}$  (4.8) is measured on the boundary of the asymptotically dS spacetime, namely at the future infinity ( $\mathcal{I}^+$ ), which is outside the cosmological horizon. Similar to the black hole horizon, we can obtain the Hawking temperature of the cosmological horizon

$$\tilde{T} = \frac{1}{4\pi r_c} \left( -(n-2)k - \left( \frac{4\beta^2}{n-1} - \frac{n}{l^2} \right) r_c^2 + \frac{2\sqrt{2}\beta}{(n-1)r_c^{n-3}} \sqrt{2\beta^2 r_c^{2n-2} + (n-1)(n-2)q^2} \right). \quad (4.10)$$

In general, the Hawking temperature of cosmological horizon is not equal to that of black hole horizon. Therefore the spacetime describes a black hole in de Sitter space is unstable quantum mechanically. The chemical potential conjugate to the charge  $Q$  is defined as

$$\tilde{\Phi} = -\sqrt{\frac{n-1}{2(n-2)} \frac{16\pi q}{r_c^{n-2}}} {}_2F_1\left[\frac{n-2}{2n-2}, \frac{1}{2}, \frac{3n-4}{2n-2}, -\frac{(n-1)(n-2)q^2}{2\beta^2 r_c^{2n-2}}\right]. \quad (4.11)$$

And the entropy of the cosmological horizon obeys the area formula

$$\tilde{S} = 4\pi\omega_{n-1}r_c^{n-1}. \quad (4.12)$$

For such defined thermodynamic quantities associated with the cosmological horizon, we can see that they indeed satisfy the first law of thermodynamics

$$d\tilde{M} = \tilde{T}d\tilde{S} + \tilde{\Phi}dQ. \quad (4.13)$$

Therefore it is seemingly reasonable to consider black hole horizon and cosmological horizon as two separate thermodynamic systems [20].

For the cosmological horizon, the Casimir energy turns out to be [15, 16]

$$\tilde{E}_c = n\tilde{E} - (n-1)\tilde{T}\tilde{S} - (n-1)\tilde{\Phi}Q = -2k(n-1)r_c^{n-2}\omega_{n-1}, \quad (4.14)$$

where  $\tilde{E} = \tilde{M}$ . Similarly we can calculate the energy of Born-Infeld electromagnetic field outside the cosmological horizon

$$\begin{aligned} \tilde{E}_q &\equiv -\omega_{n-1} \int_{r_c}^{\infty} \bar{T}_0^0 r^{n-1} dr \\ &= -\omega_{n-1} \left( \frac{4\beta^2}{n} r_c^n - \frac{2\sqrt{2}\beta r_c}{n} \sqrt{2\beta^2 r_c^{2n-2} + (n-1)(n-2)q^2} \right. \\ &\quad \left. + \frac{2(n-1)^2 q^2}{nr_c^{n-2}} {}_2F_1\left[\frac{n-2}{2(n-1)}, \frac{1}{2}, \frac{3n-4}{2(n-1)}, -\frac{(n-1)(n-2)q^2}{2\beta^2 r_c^{2n-2}}\right] \right). \end{aligned} \quad (4.15)$$



Then we find that as the case of black hole horizon, the entropy of cosmological horizon can be reexpressed in terms of the Cardy-Verlinde formula

$$\tilde{S} = \frac{2\pi l}{n-1} \sqrt{|\tilde{E}_c/k|(2(\tilde{E} - \tilde{E}_q) - \tilde{E}_c)}. \quad (4.16)$$

Here the extensive part of the energy

$$2(\tilde{E} - \tilde{E}_q) - \tilde{E}_c = 2(n-1)\omega_{n-1}r_c^2/l^2. \quad (4.17)$$

Thus we show that both the entropies associated with black hole horizon and cosmological horizon of the Born-Infeld black holes in dS space can be formally recast to a form of the Cardy-Verlinde formula, although the Casimir energy is negative in some cases, or the extensive part of energy is negative in others. These results imply that thermodynamics associated with black hole (cosmological) horizon is indeed related to that of a some CFT.

## V. FACTORIZED SOLUTION

In Sec. II we mentioned that for the static electromagnetic field solution (2.10), we have  $R''(r) = 0$  from the Einstein equations of motion, and only the solution of  $R = r$  was considered there, which gives the Born-Infeld black hole solutions in (A)dS spaces (2.13). In this section we discuss the case  $R = a$  with  $a$  being a positive constant. In this case, the metric (2.6) is a direct product of two subspaces, in which  $ds_2^2 = a^2 h_{ij} dx^i dx^j$  is an  $(n-1)$ -dimensional constant curvature space.

From the equation (2.4) of the Born-Infeld electromagnetic field, we have

$$F^{rt} = \frac{\sqrt{(n-1)(n-2)}\beta q}{\sqrt{2\beta^2 a^{2n-2} + (n-1)(n-2)q^2}} \equiv f, \quad (5.1)$$

where  $q$  is still an integration constant, which can be related to the electric charge of the solution via (2.11). Note that here the electric field (5.1) is a constant, independent of the radial coordinate  $r$ .

Solving (2.7) or (2.8) gives

$$V = -\Lambda_0 r^2 + c_1 r + c_0, \quad (5.2)$$

where  $c_1$  and  $c_0$  are two integration constants. Without loss of generality, we can take  $c_1 = 0$  through redefining the radial coordinate  $r$ . And  $c_0$  can be normalized to be one by rescaling the coordinates  $t$  and  $r$ . Then we have

$$V = 1 - \Lambda_0 r^2. \quad (5.3)$$

The constant  $\Lambda_0$  is

$$\Lambda_0 = \frac{8}{n-1}\Lambda - \frac{16\beta^2}{n-1}(1 - \sqrt{1 - f^2/\beta^2}) - \frac{8(n-3)f^2}{(n-1)\sqrt{1 - f^2/\beta^2}}. \quad (5.4)$$

From (2.9) we obtain a constraint

$$\frac{n-2}{a^2}k - \frac{2}{n-1}\Lambda + \frac{4\beta^2}{n-1} - \frac{4\beta^2}{(n-1)\sqrt{1 - f^2/\beta^2}} = 0. \quad (5.5)$$

Therefore we see that the coordinates  $t$  and  $r$  in (2.6) now span a two-dimensional constant curvature spacetime. The curvature is determined by  $\Lambda_0$  (dS if  $\Lambda_0 > 0$ , AdS if  $\Lambda_0 < 0$ ). In addition, once the parameters, the curvature parameter  $k$ , charge  $q$ , the cosmological constant  $\Lambda$ , and the Born-Infeld parameter  $\beta$  are given, then the curvature radius  $a$  of the constant curvature space  $ds_2^2 = a^2 h_{ij} dx^i dx^j$  can be determined via the relation (5.5). Finally we obtain an exact solution (2.6), which is a direct product of a two-dimensional dS (if  $\Lambda_0 > 0$ ) or AdS (if  $\Lambda_0 < 0$ ) space and an  $(n-1)$ -dimensional constant curvature space with a curvature radius  $a$ .

## VI. CONCLUSION

We have found black holes solutions in the Einstein-Born-Infeld theory with a cosmological constant in higher dimensions. These black hole solutions are asymptotically de Sitter or anti-de Sitter depending on the cosmological constant. Black hole horizon or cosmological horizon in these solutions can be a positive, zero or negative constant curvature hypersurface. Therefore these solutions generalize some exact solutions of the Einstein-Born-Infeld theory existing in the literature. We have studied thermodynamics of black hole horizon and of cosmological horizon. For the Born-Infeld black holes in AdS spaces, when the horizon is a zero or negative constant curvature hypersurface, they are always thermodynamical stable with positive heat capacity; when the horizon is a positive constant curvature hypersurface, there is a critical value on the Born-Infeld parameter for a fixed charge, above which the black hole is also thermodynamical stable. However, the Born-Infeld parameter is less than the critical value, a unstable black hole phase will appear. In addition, it has been found that although the electromagnetics of the Born-Infeld theory is non-linear, the entropies of black hole horizon and cosmological horizon can all be recast to a form of the Cardy-Verlinde formula, which indicates that the Cardy-Verlinde formula is of universality in some sense. Note that if there are some higher derivative terms of curvature, black hole entropy in these gravitational theories can not be expressed in terms of the Cardy-Verlinde formula [11]. Therefore our results implies that higher order derivative terms of electromagnetic field do not destroy the application of the Cardy-Verlinde formula. In other words, the spacetime of the Born-Infeld black hole at least in AdS space is still dual to a some CFT residing on the boundary of the AdS space. The conformal invariance of the dual CFT is not broken.

In the Einstein-Born-Infeld theory with a cosmological constant in  $D(\geq 4)$  dimensions, we have also found a factorized solution, whose spacetime is a direct product of two constant curvature spaces: one is a two-dimensional dS or AdS space and the other is a  $(D - 2)$ -dimensional positive, zero or negative constant curvature space with a constant curvature radius.

## Acknowledgements

DWP thanks Hong-Sheng Zhang, Zong-Kuan Guo, Wei He, Jian-Huang She, Qi Guo and Hao Wei for useful discussions and kind help. RGC would like to express his gratitude to the Physics Department, Baylor University for its hospitality. This work was supported by Baylor University, a grant from Chinese Academy of Sciences, a grant from NSFC, China (No. 13325525), and a grant from the Ministry of Science and Technology of China (No. TG1999075401).

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